Active Noise Control: Fundamentals and Recent Advances

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Part 1
Fundamentals
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Part 2
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Part I – Fundamentals

• Noise Pollution Problem
• ANC History
• Physical Concepts
• ANC Systems Classification
• ANC Adaptive Algorithms
Modern technologies cause increase in environmental acoustic noise because of

- Using industrial equipment such as engines, blower, fans, transformers compressors and etc.
- Growth of high density housing.
- Using lighter material in houses, cars and etc.
Chronic exposure to noise may affect the nervous system and leading to the following adverse health manifestations:

- Physiological instability
- Increased incidence of coronary artery disease
- Excessive expression of adrenaline
- Increased heart rate and cardiovascular diseases
- Constriction of blood vessels and vision impairment Seizures
Acoustic Noise Problem

Solution?

Solution 1: Turning off the noise source e.g. refrigerators, compressors, engines, ..... etc.

Solution 2: Blocking the noise!
Passive Noise Control

Solution 3: Dimming the noise!
Active Noise Control

Noise Pollution Problem

Unrealistic!!

Costly but effective for high frequency

Complicated but potentially applicable and efficient for low frequency
Passive Noise Control

Passive: no energy required to control the noise

Primary Noise

Noise suppressor

Drawbacks of passive noise control systems:
1- Bulky
2- Costly
3- Inefficient for low frequency noise
Passive Noise Control

The traditional approach to noise control uses passive techniques by using silencers – we call this approach passive because it dims the noise without consuming energy.

Gun silencer:

Engine silencer:
Passive Noise Control

- **Reactive Silencers**
  - Use the concept of impedance change caused by a combination of baffles and tubes.
  - Usually used for combustion engines.

- **Resistive Silencers**
  - Use the concept of energy loss caused by sound propagation in a duct lined with sound absorbing material.
  - Usually used for ducts propagating fan noise.
Passive Noise Control

Side-branch Resonator

- Effective for attenuating pure tone sound emitted from rotary machines in a duct.
- It is a tuned piping arrangement placed off the main piping run.
- The tuned arrangement is a closed section of pipe with a length equal to $\lambda/4$ connected to the pipe via a smaller orifice section "Helmholtz resonator".
- The aim of the resonator is to offer a negligible impedance to the propagating sound.
- If the resonator is perfect (impedance is zero), then all of the acoustic energy will flow into the resonator and none will continue down the duct.
Passive Noise Control

Expansion Chamber

The resonator here is an expansion chamber which can attenuate the sound over a wider range of frequencies than the side chamber resonator. The capability of the expansion chamber to attenuate the engine sound at low frequencies is controlled by the resonance of the volume/exhaust tube combination.
Inside a Muffler

Located inside the muffler is a set of tubes. These tubes are designed to create reflected waves that interfere with each other or cancel each other out. Take a look at the inside of this muffler:

The exhaust gases and the sound waves enter through the center tube. They bounce off the back wall of the muffler and are reflected through a hole into the main body of the muffler. They pass through a set of holes into another chamber, where they turn and go out the last pipe and leave the muffler.

http://www.howstuffworks.com/muffler.htm
Passive Noise Control

- Simple mechanise
- Efficient in global control of noise

Weaknesses:
- Bulky
- Costly
- Inefficient for low frequency noise

Example:
Find the size of the efficient silencer at
\[ f = 5\text{KHz and } f = 200\text{Hz}, \text{ where } c \approx 342\text{m/s} \]

\[ \lambda = \frac{c}{f} \]
\[ f = 5000 \text{ Hz} \Rightarrow \lambda \approx 6.8 \text{ cm} \]
\[ f = 200 \text{ Hz} \Rightarrow \lambda \approx 171 \text{ cm} \]
Active Noise Control

area under noise control

noise

anti-noise

residual noise
Active Noise Control

Strengths:
- Compact size
- Cost effective
- Efficient for low frequency noise

Weaknesses:
- A modern electro-acoustic control system required
- Only local control (inefficient in global control)
- Inefficient for high frequency noise
### Active vs Passive Noise Control

<table>
<thead>
<tr>
<th>Strengths</th>
<th>Passive</th>
<th>Active</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Simple mechanise</td>
<td>Compact size</td>
</tr>
<tr>
<td></td>
<td>Efficient in global control</td>
<td>Cost effective</td>
</tr>
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<td>Bulky</td>
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<tr>
<td></td>
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</tr>
<tr>
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<td>A modern control system required</td>
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<td></td>
<td></td>
<td>(reachable using DSP techniques)</td>
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<td></td>
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<td>Only local control</td>
</tr>
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</tbody>
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1930’s: Generating ANC Idea

M: Microphone
L: Loudspeaker
V: Controller

- Original drawings used by Paul Leug, a German engineer, in the first patent on ANC in 1936: How two sound waves can cancel each other in a short duct.

- He could not realize his idea because the available technology did not make the realization of such system possible in that time.
Active Noise Control

1950‘s : Emerging Analog ANC Devices

• An American engineer, Olson used early analog electronic technology to invent the first ANC device.

• He called his invention the “electronic sound absorber”.

• Also, he proposed some modern applications of ANC.

• More devices were invented by Fogel, Simshauser and Bose.

• Basically, these devices are analog amplifiers; they are non-adaptive and sensitive to any changes.

• These problems cannot be solved by using analog electronic technology.
Active Noise Control

ANC History

1970’s: Adaptive Noise Cancellation Using DSP

• Widrow revolutionized adaptive noise cancellation using DSP technology.

• He introduced new applications of adaptive noise cancelations in Electrocardiography and etc.

• Adaptive noise cancellation removes noise from electrical signals in electric domain (different from ANC). However, the methods used here is a base for ANC methods.
Active Noise Control

1980’s: Emerging Digital ANC Devices

- Burgess proposed the first digital ANC device for a controlled environment (acoustic duct).

- He used the ideas introduced by Widrow for adaptive noise cancellation, but he could cancel the noise in physical (acoustic) domain not in the control (electrical) domain.

- Later, Warnaka implemented the first digital ANC device for acoustic ducts.
Active Noise Control

1990’s: Developing ANC Theory

• Several researchers worked on the theory of Active Noise Control, including Morgan, Eriksson, Kuo, Elliotott and etc.

• Several methodologies, architectures and algorithms were developed.

• Powerful computing platforms for real-time control and signal processing were commercialized.

ANC History

1. Multichannel ANC
2. Feed-forward ANC
3. Feedback ANC
4. Hybrid ANC

FPGA Boards

Real-time Controllers
Active Noise Control

2000’s: Modern Application of Digital ANC

- Digital Active Noise Control were integrated into many digital devices
- Also, it can be used to enhance other technologies
Active Noise Control

Our Active Noise Control Research

- We started a comprehensive theoretical analysis on adaptive ANC.
- We developed new architecture and algorithms.
- We implemented ANC systems for acoustic duct and anechoic rooms.
- Currently, we are working on 3D ANC for creating 3D zones.
- We intend working on integrating ANC into hearing aid devices by using remote acoustic sensing.
Acoustic Wave Propagation

Direction of Sound Travel

Sound Source

Compression  Dilation  Compression  Dilation  Compression
Acoustic Wave Propagation

Sound Source

Cancelling Source
Acoustic Wave Propagation

Think about this scenario [14]
If two sound sources operate in free space and produce a series of sound waves, then at some points in space the waves cancel and at some points in space they add up.

What exactly really happens?
Adding a second sound source means that acoustic power in the environment has actually increased, specially if the sound sources are separated by a long distance. Intuitively two sound sources pumping out acoustic energy implies the sound levels have increased, not decreased!!!

This contradicts what we are aiming for though!!
Acoustic Wave Propagation

How can ever end up with global sound attenuation when active noise control techniques are employed?

This could be achieved if the total energy flow is reduced. In such case the second loudspeaker must:

1. cause a reduction in the acoustic power output of both sound sources, such that the total power is less than the power output of the original source operating alone.

2. The sound sources must resonate and when one speaker delivers energy flow to the acoustic environment the other sound source absorbs power (energy flow into the loudspeaker, not out of the loudspeaker).

As a side related thought, the actual amount of energy associated with an acoustic wave is very small, typically, a fraction of a watt. Therefore, the amount of energy being absorbed is also very small. Physically, the absorbed energy ends up helping to actually move the cone against the mechanical impedance associated with the diaphragm and suspension, the magnetic stiffness and damping, etc.
Acoustic Wave Propagation

The scenario introduced in the previous slide imposes the following constraints:

1. The distance between the sound sources must be small. From physics, the acoustic wave travels in free space omnidirectional and it does not travel directional from one loudspeaker to another. From an energy point of view, this means that a little bit of the displaced volume of air will introduce unwanted acoustic power. Thus, sound sources must be located in close proximity for energy flow to be significantly reduced. However, for a free-space acoustic radiation absorbing power by the control source often induces more energy flow out of the source of unwanted noise than it could absorb. The end result is often an increase in energy flow, not a decrease. This is a major issue in open space ANC.
2. The two sound sources must be coherent. For the flow of acoustic energy to be reduced on average, the pressure must be canceled in front of the loudspeaker cone on average. This means that one loudspeaker will have to produce a sound field which is a mirror image of that produced by the other. Every time one has an increase in pressure, the other must have a similar-sized decrease in pressure, and vice versa. This means that a control loudspeaker "knows" what the other is going to do and can act accordingly. Technically, we say that the sound sources must be coherent.
3. The sound sources must roughly be of the same size. Thus we can’t use a small loudspeaker to effectively control sound from a big sound source. The small loudspeaker simply won’t be able to cancel the pressure over a large enough region. If we confronted with big noise generator system (such as a transformer) we may use multiple loudspeakers distributed in close proximity to the unwanted acoustic source generator.
Acoustic Wave Propagation

How close do the sound sources have to be?

Intuitively, the control source must be able to duplicate the shape of the sound field generated by the primary noise source, as defined by the amplitude of the sound pressure at all points. In this way, the phase can simply be inverted and cancellation will be possible at most points in space. In the figures below, the sound field patterns of two sources quickly become "different" as the noise sources are separated.

Small separation

One wavelength separation [14]
Acoustic Wave Propagation

The effect of separation distance is quantified relative to the wavelength of the sound: \( \lambda(m) = \frac{343}{f(\text{Hz})} \)

A useful result to remember from this curve is that the separation distance should be less than one-tenth of the wavelength to achieve 10 dB power attenuation.
Acoustic Wave Propagation

A harmonics rich signal with 50Hz fundamental frequency is emitted from the source. Another signal is emitted from the control speaker (Cancellation) to cancel out some harmonics from the source.

For source signal $f = 50\text{Hz}$
$$\lambda = \frac{c}{f} = 343 \text{ (m/sec) / 50 Hz} = 6.86 \text{ m}$$
$$\frac{L}{\lambda} = 343 \text{ mm} / 6.86 \text{ m} = 0.05$$

According to the previous slide figure, the attenuation is excellent for this frequency.

For source signal $f = 100\text{Hz}$
$$\frac{L}{\lambda} = 343 \text{ mm} / 3.43 \text{ m} = 0.1$$

which also well attenuated

For source signal $f = 500\text{Hz}$
$$\frac{L}{\lambda} = 0.5$$

There is no attenuation for this frequency and all greater harmonics
Acoustic Wave Propagation

Thoughts on the Primary Noise Source and Control Source Outputs

Completely canceling the sound pressure is not really possible in the free space environment. This is due to the "spherical spreading" phenomena of waves as they move away from a source. Sound waves spread out in a circle (wave fronts) at each azimuth as they move away from a source.

*According to this scenario it would be impossible to persist the coherence at all points in space and introduce a global quiet zone.*

Each wavefront has a certain amount of energy. As the wavefront expands in diameter when moving away from the source, the energy will decrease proportionally.
Acoustic Wave Propagation

Thoughts on the Primary Noise Source and Control Source Outputs (cont.)

For example, the sound pressure amplitude at 1 meter from a sound source is twice (6 dB more than) the amplitude at 2 meters from the source.

This indicates that cancelling the signal in front of the source needs higher energy signal than the source energy to be fed to the environment by the control source if the two sources are located at different locations. *This might increase the noise in other spots.*

A sensible scenario though is to try to adjust the power and phase of the control source such that the sound pressure in front of it is mostly cancelled. This would reduce the noise in front of the control source if the environment is constrained.

*This is the scenario adopted in reducing the noise in ducts*
Acoustic Wave Propagation

Where to Locate the Error Microphone?
There is no specific answer we can find to this important question as it basically depends on the geometry of the space and many other variability. Yet, a few rules of thumb which can be used to guide microphone placement:

1. Never locate the microphone too close to the primary noise source. This inevitably leads to a control source output that is too large, and often controller saturation in response to trying to satisfy the requirement of cancellation at the microphone location.

2. Do not locate the microphone too close to the control source if global sound attenuation is desired. If the microphone is too close to the control source, then the control source output will be too low to provide cancellation away from the microphone position.

3. If multiple error microphones are used, avoid too much symmetry in placement. Often a random placement will work best.

4. The sensitivity of microphone placement tends to reduce as the control and primary source separation distance reduces.
Acoustic Wave Propagation

Causality Issue!

• In adaptive feedforward active noise control systems, a reference signal must be provided to the controller as a measure of the impending disturbance.

• If the unwanted noise is periodic, such as that coming from rotating parts, then the reference signal can be taken from a measure of the machine rotation.

Periodic signal

A point can be predicted anytime in future
Acoustic Wave Propagation

Causality Issue! (cont.)

• If the disturbance is periodic and deterministic then the reference signal can be used to predict the characteristics of the sound field at any time in future. This greatly simplifies the control system requirements, as it is not necessary to consider the timing of the reference signal relative to the unwanted sound field; it will *always* be a good predictor of the sound field.

• This type of control arrangement, is referred to as *noncausal*. The reference signal *does not have to be* the precise "cause" of the section of sound field that will be canceled out by the controller!
Acoustic Wave Propagation

Causal Signal

- The practical signal in active noise control to deal with is what is called *causal*.

- For active noise control to cancel out a given section of a sound field, the corresponding section of the reference signal must be measured and processed, and the canceling signal fed out at precisely the right time. The pairing of reference signal and unwanted noise is critical.

- The reference signal *must be* the precise cause of the section of sound field is needed to be cancelled.
Acoustic Wave Propagation

Implementing a causal active noise control system for free space wave propagation needs to carefully consider the time delay for a wave to travel between the primary source sensor and the control sensor.

This delay should match or longer than all the delays required by a signal to pass through a digital control system which comprises:

- All the delays associated with input and output filtering, sampling, and calculation of results. Also, include the delay associated with driving a loudspeaker, as it takes a finite amount of time for a loudspeaker to produce sound once an electrical signal is fed to the loudspeaker.

- These delays are dependent upon a number of physical variables, such as loudspeaker size and filter cutoff frequency, and are frequency dependent. Typical values of delay are of the order of several milliseconds or more.
Acoustic Wave Propagation

This delay comprises: (cont.)

- Sound waves travel through space at 343 m/s. During the delay of several milliseconds, the sound wave would have traveled 1 meter, 2 meters, or even more.
- To implement a causal active noise control system, a reference signal must be taken from the target noise disturbance several milliseconds before it arrives to the control sensor.
- This means the reference microphone must be located upstream of the loudspeaker by 1 meter, 2 meters, or more, so that the unwanted noise disturbance is arriving at the loudspeaker just as the canceling sound wave is being generated.
Acoustic Wave Propagation

Causality and Free Space ANC

- We indicated that for global noise control to be possible in free space the control source and primary source must be situated in close proximity, preferably less than one-tenth of a wavelength. If we assume that the overall delay in the control system path is of the order of 6 milliseconds, then:

  \[ \text{Time of wave travel} = 0.006 \text{ s} \times 343 \text{ m/s} \approx 2 \text{ meters} \]

- Then for a causal control system the sound sources must be separated by about 2 meters.
Acoustic Wave Propagation

Causality and Free Space ANC (cont.)

• So for a good global attenuation we will target frequency components where one-tenth wavelength is of the order of 2 meters.

\[ 0.1 \times \lambda(m) = 0.1 \times 343/f(Hz) \approx 2m \quad \text{and} \]

\[ f(Hz) \approx 34.3 / 2 = 17.15 \text{ Hz} \]

• To increase the targeted frequency we have to reduce the distance which is good from the coherence perspective but needs faster processing to cope with shorter delay.

We may conclude that due to the above physical constraints active noise control in free space applications is mostly limited to periodic noise applications due to the causality problem. Sample applications could be: electrical transformers, motors, fans, and machines with rotating parts.
Sound Propagation

• Sound propagation system:

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p = f$$

• $f$ is the source function.

• $p$ is the sound pressure field that is a function of time and location.

• Therefore, a sound field is the response of the sound propagation system to a sound source.
Superposition of Sound Fields

- In system theory, the response of a linear system to N individual sources, can be formulated as the summation of the system’s responses to individual sources.

- Sound propagation is a linear system; therefore, it is a subject of superposition principle.
Superposition of Sound Fields

\[ \nabla^2 p_1 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p_1 = f_1 \]
\[ \nabla^2 p_2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p_2 = f_2 \]
\[ \cdots \]
\[ \nabla^2 p_N - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p_N = f_N \]

\[ p = p_1 + p_2 + \cdots + p_N \]

\( p_1, p_2, \ldots, p_N \) are not accessible. The net sound pressure \( p \) is accessible.
Superposition of Sound Fields

ANC  →  Superposition of Noise and Anti-noise Fields

- $P_d$ represents the net sound pressure caused by all the available noise sources.

- Now, we introduce a control source to the environment, called the anti-noise source. The sound pressure caused by the control source is presented by $P_c$.

- In this situation, the net sound pressure in the environment can be formulated by the summation of $P_d$ and $P_c$.

- We can make $p=0$ by driving a proper anti-phase control source.

$$p_d = p_1 + p_2 + \ldots + p_N$$

$$p = p_d + p_c$$

$$p = 0 \iff p_c = -p_d$$
Introducing Quiet Zone

- Technically, we cannot make $p=0$ globally. However we can $p=0$ in a number of discrete points. (a single point in single channel ANC or multiple point in multi-channel ANC)

- A zone of quiet is formed around the point of interest.

- ANC aims at decreasing the sound pressure at this point as much as possible.

- A microphone, called the error microphone, has to be placed at this point. The measured signal is called the error signal: $e(n)$
ANC Model in Physical Layer
Primary Noise

• There might be several noise sources in the environment.

• The net primary noise at the quiet zone is the summation of \(d_1(n), d_2(n), \) and \(d_3(n)\).
ANC Model in Physical Layer
Primary Noise

- It is assumed that the noise field is generated by a reference noise signal $x(n)$, taking into account all the available noise sources.

- The reference noise travels across the medium to the quiet zone, causing the net primary noise $d(n)$. 
ANC Model in Physical Layer
Secondary Noise

- Anti-noise is generated by the control source.
- The sound pressure caused by the anti-noise at the quiet zone is called the secondary noise, $d'(n)$. 

Reference Noise $x(n)$

Anti-Noise $y(n)$
ANC Model in Physical Layer System Operation

- The influences of the medium on the sound from \( x(n) \) to the Quiet Zone and from \( y(n) \) to the Quiet Zone are modelled by the primary path \( P \) and secondary path \( S \) respectively.
Quiet zone complexity

1. Single-channel
Only one error microphone is used.
• Compact and simple design.
• Limited quiet zones.

2. Multi-channel
The error criteria is based on measurements collected from several microphones located in proximity via several channels
• Extended quiet zones.
• Expensive and complicated design.
Control Paradigms

1. Feed-forward
An upstream microphone (reference microphone) is used to measure the noise in a reference location.
- Good stability and robustness
- Requires an extra microphone

2. Feedback
An internal model estimates a reference signal using signals $e(n)$ and $y(n)$ that are available for ANC.
- Compact design
- Only for predictable noise (periodic)
Feed-forward Architecture

- The control system can measure $x(n)$ and $e(n)$
- $y(n)$ is produced as the response of a linear filter to $x(n)$
- The Linear filter is adjusted to minimize $e(n)$
Feed-forward Architecture
Feed-forward Architecture

Duct

Acoustic Domain

Electrical Domain

Reference Mic

Control Source

Error Mic

Antialiasing Filter

Smoothing Filter

Antialiasing Filter

Sample/Hold

Sample/Hold

Sample/Hold

ADC

DAC

ADC

Adaptive Controller (Processor, DSP, FPGA, Controller)
Feed-forward Adaptive Controller Architecture

- Reference Mic
- Control Source
- Error Mic

1. Digital Filter
2. Adaptive Algorithm
3. Secondary Path Model
4. Adaptive Controller

\[ \Sigma \]
• W is a linear Adaptive Digital Filter (ADF)
• P and S are the primary and secondary paths.
• $x(n)$: reference noise measured by the reference microphone.
• $d(n)$: primary noise at the quiet zone.
• $y(n)$: ant-noise generated by the control system.
• $d'(n)$: secondary noise.
• $e(n)$: error noise measured by the error microphone.
Feed-forward Architecture

\[ D(z) = P(z)X(z) \]
\[ D'(z) = W(z)S(z)X(z) \]
\[ E(z) = D(z) + D'(z); \text{ thus} \]
\[ E(z) = P(z)X(z) + W(z)S(z)X(z) \]
\[ \text{for } E(z) = 0, \text{ we should have} \]
\[ W(z) = W_0(z) = \frac{-P(z)}{S(z)} \]
\( W_0(z) = -\frac{P(z)}{S(z)} \) gives an optimal feed-forward ANC controller.

- \( P(z) \) and \( S(z) \) are unknown.
- \( S(z) \) may have non-minimum phase zeros or it may have zeros at the origin.
- ANC algorithm must reach an stable and causal estimate of \( W_0(z) \).
Feed-forward Architecture Challenges

Example-1: Causality Challenge

- \( p(n) = \delta(n - 5) - 0.4 \delta(n - 6) + 0.2 \delta(n - 7) \)
- \( s(n) = \delta(n - 8) - 0.6 \delta(n - 9) \)

\( W_0(z) \) is not causal because its output value at time \( n \) depends on its future values.

For having causal solution, the delay in \( P \) should be greater than \( S \).

\[
P(z) = z^{-5} - 0.4z^{-6} + 0.2z^{-7}
\]
\[
S(z) = z^{-8} - 0.6z^{-9}
\]
\[
W_0(z) = -\frac{P(z)}{S(z)}
\]
\[
W_0(z) = -\frac{z^{-5} - 0.4z^{-6} + 0.2z^{-7}}{z^{-8} - 0.6z^{-9}}
\]
\[
W_0(z) = -\frac{z^3(1 - 0.4z^{-1} + 0.2z^{-2})}{1 - 0.6z^{-1}}
\]
Example 2: Stability Challenge

\[ p(n) = \delta(n - 5) - 0.4 \, \delta(n - 6) + 0.2 \, \delta(n - 7) \]
\[ s(n) = \delta(n - 2) - 0.7 \, \delta(n - 3) - 0.6 \, \delta(n - 4) \]

\[ P(z) = z^{-5} - 0.4z^{-6} + 0.2z^{-7} \]
\[ S(z) = z^{-2} - 0.7z^{-3} - 0.6z^{-4} \]

\[ W_o(z) = -\frac{P(z)}{S(z)} \]
\[ W_o(z) = -\frac{z^{-5}-0.4z^{-6}+0.2z^{-7}}{z^{-2}-0.7z^{-3}-0.6z^{-4}} \]
\[ W_o(z) = -\frac{z^{-3}(1-0.4z^{-1}+0.2z^{-2})}{1-0.7z^{-1}-0.6z^{-2}} \]

P(z) = 0 : Zeros located @ z = 0.2±0.4j
S(z) = 0 : Poles located @ z = -1.2 and z = 0.5

We can reach a stable estimate of \( W_o(z) \) provided that:

The delay in \( P \) is greater than the delay in \( S \)
• The control system can only measure $e(n)$ and $x(n)$ has to be estimated from $e(n)$
• $y(n)$ is produced as the response of a linear filter to the estimate of $x(n)$
• The Linear filter is adjusted to minimize $e(n)$
• An ideal reference signal should have all the information of the primary noise; thus, it is identical to the primary noise in ideal case: $p(n) = 1$
• $W$ is a linear Adaptive Digital Filter (ADF) that is an adjustable digital filter.
• $S$ is the secondary path.
• $x(n) = d(n)$: primary noise at the quiet zone (must be identified)
• $y(n)$: anti-noise generated by the control system.
• $d'(n)$: secondary noise.
• $e(n)$: error noise measured by the error microphone.
• An estimate of $d(n)$ can be reached by using an internal model of the system.
• $e(n)$ is available by using the error microphone; $e(n) = d(n) + s(n) \ast y(n)$
• $y(n)$ is available for the control system (because it is an electrical signal)
• If an estimate of $s(n)$ is available; then $d(n)$ can be estimated by

$$d(n) = e(n)^2 \cdot s(n) \ast y(n)$$
\[ E'(z) = W(z)S(z)X(z) \]

Since \( X(z) \approx D(z) \),

\[ D'(z) = W(z)S(z)D(z) \]

\[ E(z) = D(z) + D'(z); \text{ thus} \]

\[ E(z) = D(z) + W(z)S(z)D(z) \]

for \( E(z) = 0 \), we should have

\[ W_0(z) = -\frac{1}{S(z)} \]
$W_0(z) = -\frac{1}{s(z)}$ gives an optimal feedback ANC controller.

But $S(z)$ is unknown.

$S(z)$ may have non-minimum phase zeros and it always has zeros at the origin (due to delay in signal channel).

ANC algorithm must reach an stable and causal estimate of $W_0(z)$.

Feedback ANC is an special case of Feed-forward ANC where $P = 1$. 

**ANC Systems Classification**
Internal Model Accuracy Challenge:

\[ X = E - S_{Model}Y \]

\[ Y = WX \]

\[ Y = W(E - S_{Model}Y) \]

\[ \Rightarrow (1 + WS_{Model})Y = WE \]

\[ \Rightarrow Y = \frac{W}{1 + WS_{Model}}E \]

\[ E = D + SY \]

\[ E = D + \frac{WS}{1 + WS_{Model}}E \]

\[ \Rightarrow \left(1 - \frac{WS}{1 + WS_{Model}}\right)E = D \]

\[ E = D + \frac{WS}{1 + WS_{Model}}E \]

\[ \Rightarrow E = \frac{1}{1 - \frac{WS}{1 + WS_{Model}}}D \]

\[ \Rightarrow E = \frac{1 + WS_{Model}}{1 + W(S_{Model} - S)}D \]

\( S_{Model} - S \) can affect on the stability and dynamics of the system.
Causality Challenge (Example 1):

\[ s(n) = \delta(n - 8) - 0.6 \delta(n - 9) \]

\[ w_0(n) = -s^{-1}(n) \Rightarrow W_0(z) = -\frac{1}{S(z)} \]

\[ \Rightarrow S(z) = z^{-8} - 0.6z^{-9} \]

\[ w_0(z) = -\frac{1}{z^{-8} - 0.6z^{-9}} \]

\[ \Rightarrow W_0(z) = -\frac{z + 8}{1 - 0.6z^{-1}} \]

• \( W_0(z) \) is not causal because its output value at time \( n \) depends on its future values.

• Here, we cannot solve this causality problem similar to the feedforward architecture because the primary path is \( P=1 \).

• This issue causes the feedback architecture to be applicable only for predictable noise (narrow band noise).
Feedback Architecture Realization Challenges

Stability Challenge (Example 2):

- \( s(n) = \delta(n) - 0.7 \delta(n-1) - 0.6 \delta(n-2) \)
- \( w_o(n) = -s^{-1}(n) \Rightarrow W_o(z) = -\frac{1}{S(z)} \)

\[ S(z) = 1 - 0.7z^{-1} - 0.6z^{-2} \]
\[ W_o(z) = -\frac{1}{1-0.7z^{-1}-0.6z^{-2}} \]

\( S(z) = 0 \): Poles located @ \( z = -1.2 \) and \( z = 0.5 \)

We cannot solve this stability problem using the same logic used for the feed-forward architecture, because there is no primary path here. This issue causes feedback architecture to be less stable and more sensitive than feed-forward architecture.
In feed-forward architecture, $P$ is an arbitrary linear system.
In feedback architecture, $P=1$.
Multi-channel ANC block diagram can be reached by generalizing this block diagram.
The challenge with this diagram is: *identification of the optimal controller* $w$, while $P$ and $S$ are unknown systems.
• An adaptive algorithm is responsible for the automatic adjustment (identification) of the controller $W$.
• Usually, ANC adaptive algorithms are stochastic optimization algorithms.
• Remember that $x(n)$ and $e(n)$ are available to the control system through measurement and $y(n)$ is produced by the control system.
ANC system is similar, but not identical, to the adaptive filtering framework.

Similar to the standard adaptive filtering framework in the following sense:

1. The control system consists of a controller $W$, coupled to an adaptive algorithm.

2. $W$ produces the output signal $y(n)$ in response to a reference signal $x(n)$.

3. An adaptive algorithm is responsible for automatic adjustment of $W$ in order to minimize the power of a error signal $e(n)$. 
ANC vs Standard Adaptive Filtering

ANC Adaptive Algorithms

- In standard adaptive filtering, there is no secondary path; the optimal solution for $W = P$.
- In ANC adaptive filtering, a secondary path modifies the error signal and the optimal solution is achieved when $W = S^{-1}P$.
- This difference causes the standard adaptive filtering algorithms cannot be used efficiently in adaptive ANC; it may easily go to instability.
ANC vs Standard Adaptive Filtering

Standard Adaptive Filtering

ANC Adaptive Filtering

Standard Adaptive Digital Filtering Algorithms

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>ANC Adaptive Filtering Algorithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least Mean Square (LMS)</td>
<td>Filtered-x LMS</td>
</tr>
<tr>
<td>Recursive Least Square (RLS)</td>
<td>Filtered-x RLS</td>
</tr>
<tr>
<td>Affine Projection (AP)</td>
<td>Filtered-x AP</td>
</tr>
</tbody>
</table>

ANC Adaptive Algorithms

(differences)
LMS and FxLMS Algorithms

- LMS (Least Mean Square) is the most well known adaptive filtering algorithm. It is widely used for the adaptive identification of linear systems using transversal filters.
- FxLMS algorithm is a modified version of the LMS algorithm, that can be efficiently used in the ANC adaptive filtering framework.
- FxLMS algorithm can be generally used for any adaptive inverse control applications.
FxLMS ultimate goal is minimizing the noise (error) power. Similar to LMS and according to the method of steepest descent, the following update equation can lead to this minimization:

\[ \mathbf{w}_{\text{New}} = \mathbf{w}_{\text{Old}} + \mu \nabla J(n) \]

Where \( \mathbf{w} \) is the controller weight vector, \( \mu \) is an adaptation step-size (scalar), \( \nabla \) denotes the gradient operator (with respect to \( W \) parameters), and \( J(n) \) is the power of the error signal.
First, we need to estimate the error signal power \( J(n) \):

- By definition,

\[
J(n) = E\{e^2(n)\}
\]

Where \( E\{.\} \) denotes statistical expectation operator and

- \( E\{.\} \) is a theoretical function. To avoid this operator, \( J(n) \) is approximated by

\[
J(n) \approx e^2(n)
\]

- We can estimate \( \nabla J(n) \) as follows:

\[
\nabla J(n) = \nabla e^2(n) \Rightarrow \\
\nabla J(n) = 2e(n)\nabla e(n)
\]

- Now, we need to estimate \( \nabla e(n) \).
FxLMS Algorithm Derivation \( \nabla e(n) \)

ANC Adaptive Algorithms

\[ w_{\text{New}} = w_{\text{Old}} + \mu \nabla J(n) \]

\[ \nabla J(n) = 2e(n)\nabla e(n) \]

\[ \nabla e(n) = s(n) \ast \nabla y(n) \]

- From the block diagram,

\[ e(n) = d(n) + s(n) \ast y(n) \]

\( s(n) \) is the secondary path impulse response.

- \( \nabla e(n) \) can be estimated by

\[ \nabla e(n) = s(n) \ast \nabla y(n) \]

- Now, we need to estimate \( \nabla y(n) \).
• From the block diagram,

\[ y(n) = \mathbf{w}^T \mathbf{x}(n) \]

where \( \mathbf{w} \) is the controller weight vector and \( \mathbf{x} \) is the reference signal tap vector (of the same length as the controller length)

• Now, \( \nabla y(n) \) can be expressed by

\[ \nabla y(n) = \frac{\partial y(n)}{\partial \mathbf{w}} \Rightarrow \]

\[ \nabla y(n) = \mathbf{x}(n) \]
FxLMS Algorithm Derivation

Combining all the results:

\[ w_{\text{New}} = w_{\text{Old}} + 2\mu e(n) \cdot s(n) \ast x(n) \]

\( s(n) \) should be estimated through off-line or online secondary path techniques. If \( \hat{s}(n) \) denotes an estimate of \( s(n) \),

\[ w_{\text{New}} = w_{\text{Old}} + 2\mu e(n) \cdot \hat{s}(n) \ast x(n) \]

OR

\[ w_{\text{New}} = w_{\text{Old}} + 2\mu e(n) \cdot x_f(n) \]

This update equation is called the FxLMS algorithm.
FxLMS-based ANC Simulation using MATLAB

```matlab
>> e = FxLMS(L,N,x,d,S,Model,mu);
```

- $L = \text{Filter Length}$
- $N = \text{maximum time index}$
- $x = \text{original noise (reference signal)}$
- $d = \text{primary noise (}d = p \times x\text{)}$
- $S = \text{secondary path}$
- $\text{Model} = \text{secondary path model}$
- $\mu = \text{step-size}$

MATLAB Library

FxLMS.m
CLMS.m
FeLMS.m
LeakyFxLMS.m
NFxLMS.m
LMS.m
NLMS.m
FxLMS Algorithm Variants

ANC Adaptive Algorithms

In order to solve the problems associated with the FxLMS algorithm, several modified algorithm have been developed, some of which are introduced here:

1. Normalized FxLMS Algorithm
2. Leaky FxLMS
3. Variable Step Size FxLMS
4. Filtered Weight FxLMS
Normalized FxLMS Algorithm

• The stability of the FxLMS algorithm is highly depended on the $x_f(n)$ power. In order to make the stability behaviour independent on this factor, $x_f(n)$ can be normalized by its Euclidian norm, resulting in the Normalized FxLMS algorithm:

$$w_{\text{New}} = w_{\text{Old}} + 2\mu e(n) \frac{x_f(n)}{\|x_f(n)\|^2}$$

• Alternatively, NFxLMS can be implemented by

$$w_{\text{New}} = w_{\text{Old}} + 2\mu e(n) \frac{x_f(n)}{P(n)}$$

$$P(n) = P(n-1) - x_f^2(n-L) + x_f^2(n)$$

That involves only 2 multiplication- and 2 addition- operations for calculating $P(n)$. 
Leaky FxLMS Algorithm

ANC Adaptive Algorithms

- The FxLMS algorithm diverges if the reference signal does not contain sufficient spectral excitation. For solving this problem a leaking mechanism can be added to the FXLMS:

\[ \mathbf{w}_{\text{New}} = v\mathbf{w}_{\text{Old}} + 2\mu e(n)\mathbf{x}_f(n) \]

- Where \(0 < v \leq 1\) is called the leakage factor.
- This mechanism add low-level white noise to the reference signal.
Variable Step-Size FxLMS Algorithm

- A small $\mu$ causes slow convergence rate but high steady-state performance. On the other hand, a large $\mu$ causes fast convergence rate but low steady-state performance.

In order to cope with this trade-off, a variable step-size can be used: In the transient conditions, the step size is set to a relatively large number and it is decreased while the system converges to its steady state level.

$$\mathbf{w}_{\text{New}} = \mathbf{w}_{\text{Old}} + 2\mu(n)e(n)\mathbf{x}_f (n)$$
Filtered Weight FxLMS Algorithm

ANC Adaptive Algorithms

• Recently, we introduced a new variant of the FxLMS algorithm that gives a control over the dynamics of the adaptation process in active noise control.

• In this algorithm, the old weight vector is filtered by a recursive filter before being used by the update equation:

\[ w_{New} = f(n) \ast w_{Old} + 2\mu e(n)x_f(n) \]

where \( f(n) \) is a recursive filter adjusted in such a way that the desired dynamical behaviour is achieved.

• This algorithms shows more robustness than the other FxLMS variants, while being used in active control of sound in open space or in uncontrolled rooms.
Other ANC Adaptive Algorithms

- Basic FxLMS algorithms are simple and computationally efficient algorithms.
- However, they suffer from slow convergence rate. In order to solve this problem, more sophisticated algorithms have been introduced, such as:
  1. Filtered-U Recursive LMS Algorithm
  2. FxRLS Algorithm
  3. Frequency domain FxLMS
  4. Subband FxLMS
Filtered-U Recursive LMS Algorithm

• As mentioned earlier, the FxLMS can only adjust a transversal filter:

\[ W(z) = w_0 + w_1z^{-1} + \cdots + w_{L-1}z^{-L+1} \]

When the impulse response of the optimal controller is long, a very high order transversal filter is required. This causes, high computational cost as well as low convergence rate.

• In order to solve this problem, a recursive filter can be used:

\[ W(z) = \frac{a_0 + a_1z^{-1} + \cdots + a_{K-1}z^{-K+1}}{b_1z^{-1} + \cdots + b_Mz^{-M}} \]

• For automatic adjustment of this controller structure, Filtered U recursive LMS algorithm must be used. This algorithm can be derived based on the same logic used in the derivation of the FxLMS algorithm.
Filtered-U Recursive LMS Algorithm

$a_k$ coefficients are updated by

$$a_{New} = a_{Old} + 2\mu e(n) \cdot s(n) \cdot x(n)$$

and $b_k$ coefficients are updated

$$b_{New} = b_{Old} + 2\mu e(n) \cdot s(n) \cdot y(n - 1)$$

where $y(n - 1)$ is the previous output of the ANC controller. Alternatively, this algorithm can be formulated by

$$\begin{bmatrix} a_{New} \\ b_{New} \end{bmatrix} = \begin{bmatrix} a_{Old} \\ b_{Old} \end{bmatrix} + 2\mu e(n) \cdot s(n) \cdot \begin{bmatrix} x(n) \\ y(n - 1) \end{bmatrix}$$
FxRLS Algorithm

- FxRLS algorithm is a modified version of Recursive Least Square (RLS) algorithm, that is a super efficient adaptive algorithm; however, with a huge computation cost.

- LMS algorithm tries to adjust the filter coefficients in such a way that the most recent sample of the error signal to be minimized.

- RLS looks at the M most recent samples of the error signal.
FxRLS Algorithm

- FxRLS algorithm involves matrix computation, resulting in a higher computational cost, compared to the FxLMS algorithm.

- The FxRLS update equation is given by

\[
\dot{z}(n) = \lambda^{-1}\dot{Q}(n-1)x_f(n)
\]

\[
\dot{k}(n) = \frac{\dot{z}(n)}{x_f^T(n)\dot{z}(n)+1}
\]

\[
w_{\text{New}} = w_{\text{Old}} + \dot{k}(n)e(n-1)
\]

\[
\dot{Q}(n) = \lambda^{-1}\dot{Q}(n-1) - \dot{k}(n)\dot{z}^T(n)
\]
The computations associated with the ANC adaptation process can be performed in the frequency domain, rather than the time domain. This causes a faster and more computationally efficient computing; however, the outputs have to be re-converted to the time-domain:
Frequency Domain Algorithms

Based on this diagram, most of time-domain ANC adaptive algorithms can be formulated and implemented in the frequency domain. For example, the frequency domain FxLMS algorithm is given by

\[ W_{New} = W_{Old} + 2\mu E(n) \cdot X_f^*(n) \]

Note that, the realization of this algorithm does not require computing any convolutions.

This means:
1. No convolution in computing \( Y \)
2. No convolution in computing \( X_f \)
Sub-band Algorithms

All the computations associated with an ANC algorithm can be separated into different sub bands. This causes more efficient computation with higher convergence rate.

ANC Adaptive Algorithms
## Comparing ANC Algorithms

<table>
<thead>
<tr>
<th>Algorithm(s)</th>
<th>Strengths</th>
<th>Weaknesses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FxLMS</strong></td>
<td>Simple real-time realisation</td>
<td>Slow convergence rate for broad-band noise</td>
</tr>
<tr>
<td></td>
<td>Low computational complexity</td>
<td></td>
</tr>
<tr>
<td><strong>FxRLS or FxAP</strong></td>
<td>Fast convergence</td>
<td>Huge computational complexity</td>
</tr>
<tr>
<td></td>
<td>Low steady state residual noise</td>
<td>Difficult real-time realisation</td>
</tr>
<tr>
<td><strong>Freq. Domain FxLMS</strong></td>
<td>Low computational complexity</td>
<td>Only suitable for narrow-band noise</td>
</tr>
<tr>
<td></td>
<td>Fast convergence</td>
<td></td>
</tr>
<tr>
<td><strong>Sub-band FxLMS</strong></td>
<td>Low computational complexity</td>
<td>Only suitable for narrow-band noise</td>
</tr>
<tr>
<td></td>
<td>Fast convergence</td>
<td></td>
</tr>
</tbody>
</table>

**FxLMS-Based ANC**

APSIPA ASC 2013

97
FxLMS Performance Hierarchy

- Limit to adaptation process determined by secondary path modelling error
- Limit to adaptation process determined by adaptation step-size
- Limit to adaptation process determined by filter length
- Limit to adaptation process determined by noise bandwidth
- Limit to adaptation process determined by secondary path impulse response duration

Electronic Control System

Physical Plant

Adaptation Process Performance
Part II – Recent Advances

- ANC Modelling & Analysis
- ANC Dynamic Control
ANC Modeling & Analysis

Part II – Recent Advances
ANC Modeling & Analysis

Content

• Motivation
• Classification
• ANC Performance
• Available Formulations
• Modeling Variables
• Independence Assumptions
• Core Equations
• Models 1, 2 and 3 (Derivation and Analysis)
• Validation of Results
Motivation

We discovered that existing theoretical understanding of ANC is not able to describe ANC behaviours in real-life experiments, as well as computer simulation experiments with realistic working conditions.

ANC Theory is much behind and it has a long way to go! A way through Math, Physics and Engineering.

This is where I’d love to be!
Motivation

Understanding behaviours of ANC systems

practical

White noise generator
Motivation

Understanding behaviours of ANC systems

1. What are the stability conditions?
2. What is the SS performance?
3. What is the convergence rate?

Modelling and Analysis
**ANC Modeling & Analysis**

**Classification | Concepts**

**ANC Layout (standard)**
- Shows the exact flow of physical and control variables.
- Signals are represented by stochastic time series.

**ANC Models**
- Gives the statistics of physical and control variables.
- Signals are represented by their statistics (usually 1\textsuperscript{st} and 2\textsuperscript{nd} order).

**ANC Analysis**
- Indicates ANC behaviour and performance.
- Models are analysed either numerically or analytically.
ANC Modeling &
Analysis

Classification | Available Modelling Techniques

**Approach 1: Model Input/Outputs**

- We need both models types to fully understand ANC.
- We developed a Type 1 model based on a novel stochastic analysis.
- We developed a Type 2 model based on a novel root locus analysis.

Suitable for understanding ANC behaviour in **Physical Domain (Acoustic)**

Suitable for understanding ANC behaviour in **Control Domain (Electronics)**
Classification | Available Modelling Techniques

Approach 2: Model Formulation

1. Numerical Models
   - Give the statistics of physical and control variables.
   - Give the behaviour and performance of the ANC system.
   - Should be analysed for a particular case.

2. Analytical Models
   - Give the statistics of physical and control variables.
   - Give the behaviour and performance of the ANC system.
   - Can be used for parametric analysis.
   - Can be used for the derivation of closed-form equations formulating the behaviour of ANC systems.

- We aimed at developing only analytical models.
Classiﬁcation | Models of Interest

• Models of interest are analytical models that can be used for the parametric analysis of ANC behaviours.

• Through these models, we look for closed form expressions for formulating the three following characteristic of ANC systems.

  1. Stability
  2. Stead State (SS) Performance
  3. Convergence Rate
ANC Performance | Stability

Parameters influencing on ANC stability

**Physical Plant Parameters**

- Parameters of the noise to be cancelled:
  - Power: $\sigma_x^2$
  - Bandwidth: $B_w$
- Impulse responses of secondary path:
  \[ s = [s_0, s_1, \ldots, s_Q]^T \]

**Control System Parameters**

- Step-size $\mu$
  What is the allowed step-size range for having stable ANC?
- Secondary path model $\hat{s}$
  What is the allowed range of secondary path estimation errors?

\[
w_{New} = w_{Old} + \mu e(n) [s(n) \star x(n)]
\]
ANC Performance | Stability Measures

Step-size upper bound: $\mu_{max}$
ANC is stable when the adaptation step-size $\mu$ is a positive scalar smaller than $\mu_{max}$.

$$0 < \mu < \mu_{max}$$

Secondary path estimate maximum phase error
ANC is stable when the phases of an actual secondary path and its estimate (for a particular frequency bin) is less than a particular level $\varphi_{max}$.

$$0 < \varphi < \varphi_{max}$$

RQ1: Can we obtain a closed form formulation for $\mu_{max}$ and $\varphi_{max}$ in terms of physical plant and control system parameters?
## ANC Performance | SS Conditions

Parameters influencing on SS performance

### Physical Plant Parameters
- Parameters of the noise to be cancelled:
  - Power: $\sigma_x^2$
  - Bandwidth: $B_w$
- Impulse responses of secondary path:
  $$s = [s_0, s_1, \ldots, s_Q]^T$$

### Control System Parameters
- Step-size $\mu$
  - What are the effects of $\mu$ on SS performance?
- Secondary path model $\hat{s}$
  - What are the effects of $\hat{s}$ on SS performance?
ANC Performance | SS Measures

Minimum Achievable Residual Noise Power: $J_{\text{min}}$

$J_{\text{min}}$ is a parameter independent on the control system parameters. It can be formulated as a function of physical plant parameters:

Misadjustment Level: $M$

$M$ shows the deviation of the residual noise power from $J_{\text{min}}$ for a particular adaptive algorithms. Consequently, it is a function of both physical plant and control system parameters.

RQ2: Can we obtain a closed form formulation for $M$ in terms of physical plant and control system parameters?
ANC Performance | Convergence Rate (CR)

Parameters influencing on CR

**Physical Plant Parameters**

- Parameters of the noise to be cancelled:
  - Power: $\sigma_x^2$
  - Bandwidth: $B_w$
- Impulse responses of secondary path:
  $$\mathbf{s} = [s_0, s_1, \ldots, s_Q]^T$$

**Control System Parameters**

- Step-size $\mu$
  What are the effects of $\mu$ on CR?
- Secondary path model $\hat{\mathbf{s}}$
  What are the effects of $\hat{\mathbf{s}}$ on CR?
ANC Performance | CR Measures

No standard measures!

RQ3: Can we obtain a closed form formulation for CR in terms of physical plant and control system parameters?
ANC Performance | Analysis Objectives

- Models of interest are analytical models that can be used for the parametric analysis of ANC behaviours.
- Through these models, we look for closed form expressions for formulating the three following measures of ANC performance:
  1. Stability $\mu_{max}=? \quad \varphi_{max}=?$
  2. Stead-state Performance $M=?$
  3. Convergence Rate $CR=?$
- We aim at considering as many as physical plant and control systems parameters possible: $\sigma_x^2 \quad B_w \quad p \quad \mu \quad \hat{s}$
ANC Performance | Analysis Objectives

RQ1: a closed form formulation for $\mu_{max}$ and $\varphi_{max}$ in terms of parameters of the physical plant and control system?

$$\mu_{max}(\sigma_x^2, B_w, s, \hat{s}) = ?$$

$$\varphi_{max}(\sigma_x^2, B_w, \mu, s, \hat{s}) = ?$$

RQ2: a closed form formulation for $M$ in terms of parameters of the physical plant and control system?

$$M(\sigma_x^2, B_w, \mu, s, \hat{s}) = ?$$

RQ3: a closed form formulation for CR in terms of parameters of the physical plant and control system?

$$CR(\sigma_x^2, B_w, \mu, s, \hat{s}) = ?$$
### Available Formulations (for performance measures)

#### Most Famous Formulations

- **Elliott’s Upper Bound of Step-Size**

  \[
  \mu_{\text{max}} = \frac{1}{\sigma_x^2 ||s||^2 (L+D)}
  \]

- **Morgan’s 90° Stability Condition**

  \[
  \varphi_{\text{max}} = 90°
  \]

- **Bjarnason’s Mis-adjustment level**

  \[
  M = \frac{\mu L \sigma_x^2 ||s||^2}{2 \left(1 - \frac{\mu}{\mu_{\text{max}}} \right)}
  \]

#### Common Modelling Assumptions

- **Noise**: broad-band white signal

  \[
  B_w = 1
  \]

  ![Noise spectrum](image)

- **Secondary path**: pure-delay system

  ![Secondary Path IR](image)

- **Secondary path model**: perfect

  \[
  s = \hat{s}
  \]
Available Formulations | Propagation in secondary path

**Existing Investigations**

- Sound Propagation in the secondary path is represented by a Pure-Delay System:

**Realistic Case**

- Sound Propagation is more complex and should be represented by a FIR system.
Available Formulations | Noise Spectrum

**Existing Investigations**

- Noise is represented by either a broadband white signal (stochastic) or a harmonic signal (deterministic).

**Realistic Case**

- Noise is represented by a band-limited stochastic signal.

**Broadband white noise**

**Harmonic noise**

**Spectrum**

**BL white noise**

**Spectrum**

**B_w**

**B_w → 1**

**B_w → 0**

**Broadband white noise**

**Harmonic noise**

**Spectrum**
ANC Modeling & Analysis

Available Formulations | Secondary Path Estimate

**Existing Models**
- Perfect estimation without any errors

**Realistic Case**
- Imperfect estimation with errors

---

**Secondary Path IR (Actual)**

**Secondary Path Estimate IR**

**D**
ANC Modeling & Analysis

Modeling Variables

Original Variables:
- Input signal $x(n)$ and its tap vector:
  $$x(n) = [x(n) \ x(n - 1) \ ... \ x(n - L + 1)]^T$$
- Input signal $d(n)$
- Output Signal: $e(n)$

Original Parameters:
- Adaptation step-size $\mu$
- Secondary path IR vector and its estimate:
  $$s = [s_0 \ s_1 \ ... \ s_{Q-1}]^T$$
  $$\hat{s} = [\hat{s}_0 \ \hat{s}_1 \ ... \ \hat{s}_{Q-1}]^T$$
- Weight vector:
  $$w(n) = [w_0(n) \ w_1(n) \ ... \ w_{L-1}(n)]^T$$

$$w(n \to \infty) = w_{opt} = R^{-1}p$$

$$R = E\{x(n)x(n)^T\}$$

$$p = E\{x(n)d(n)\}$$
Modeling Variables | Rotation Matrix

\[ \mathbf{R} = E\{\mathbf{x}(n)\mathbf{x}(n)^T\} \]

Diagonalisation of \( \mathbf{R} \) results in

\[ \mathbf{R} = \mathbf{F}^T \Lambda \mathbf{F} \]

where \( \mathbf{F} \) is a modal matrix formed by the Eigen vectors of \( \mathbf{R} \)

\[ \mathbf{F} = [ \mathbf{F}_0 \ | \ \mathbf{F}_1 \ | \ \cdots \ | \ \mathbf{F}_{L-1} ] \]

and \( \Lambda \) is a diagonal matrix of Eigen values of \( \mathbf{R} \)

\[ \Lambda = \begin{bmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{L-1} \end{bmatrix} \]

\( \mathbf{F} \) has the properties of a rotation matrix:

\[ \mathbf{F} \mathbf{F}^T = \mathbf{I}_{L \times L} \]

\[ \det(\mathbf{F}) = 1 \]

Therefore, \( \mathbf{F} \) can be used a rotation matrix.
Modeling Variables | Rotated Variables

Rotated Reference Signal

\[ z(n) = F^T x(n) \]

Modeling the system in terms of \( z(n) \) is more convenient because:

<table>
<thead>
<tr>
<th>Original signal</th>
<th>Rotated signal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E{x(n)^T x(n)} )  = ( R )</td>
<td>( E{z(n)^T z(n)} )  = ( \Lambda )</td>
</tr>
<tr>
<td>( R ) is an arbitrary square matrix</td>
<td>( \Lambda ) is a diagonal matrix</td>
</tr>
</tbody>
</table>

Rotated Error Weight Vector

\[ c(n) = F^T [w(n) - w_{opt}] \]

Modeling the system in terms of \( c(n) \) is more convenient because:

<table>
<thead>
<tr>
<th>Original vector</th>
<th>Rotated vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w(\infty) = R^{-1} p )</td>
<td>( c(\infty) = 0 )</td>
</tr>
</tbody>
</table>
Independence Assumptions

Assumption 1: Independence of Reference Tap Vectors

\( z(n) \)

\( x(n) \) samples are statistically independent.

\[
E\{x(n)x(n - m)\} = 0 \text{ where } m \neq 0 \implies E\{z(n)z(n - m)\} = 0 \text{ where } m \neq 0
\]

\[
E\{x^2(n)\} = \sigma_x^2 \implies E\{x^2(n)\} = \sigma_x^2
\]

\( z(n) \)

Also, consecutive samples of \( x(n) \) are statistically independent vectors:

\[
E\{x(n)^T x(n - m)\} = \mathbf{0}_{L \times L} \text{ where } m \neq 0 \implies E\{z(n)^T z(n - m)\} = 0 \text{ where } m \neq 0
\]

\[
E\{x(n)^T x(n)\} = \mathbf{R} \implies E\{z(n)^T z(n)\} = \Lambda
\]
Independence Assumptions

Assumption 2: Independence of Optimal Error and Reference Signal

\( x(n) \) and \( e_{opt}(n) \) are statistically independent.

\[
E\{x(n_1)e_{opt}(n_2)\} = E\{x(n_1)\}E\{e_{opt}(n_2)\} = 0
\]

\[
\Rightarrow E\{z(n_1)e_{opt}(n_2)\} = E\{z(n_1)\}E\{e_{opt}(n_2)\} = 0
\]

Assumption 3: Independence of Weights and Reference Signal

\( w(n) \) and \( x(n) \) are statistically independent.

\[
E\{w(n_1)^Tx(n_2)\} = E\{w(n_1)^T\}E\{x(n_2)\} = 0
\]

\[
\Rightarrow E\{c(n_1)^Tz(n_2)\} = E\{c(n_1)^T\}E\{z(n_2)\} = 0
\]
Core Eqs. | Weight Vector Update Eq.

- \( \mathbf{w}(n + 1) =? \)

\[
\begin{align*}
\mathbf{w}(n + 1) &= \mathbf{w}(n) + \mu e(n) \sum_q \hat{s}_q \mathbf{x}(n - q) \\
\mathbf{z}(n) &= \mathbf{F}^T \mathbf{x}(n) \\
\mathbf{c}(n) &= \mathbf{F}^T \{\mathbf{w}(n) - \mathbf{w}_{opt}\}
\end{align*}
\]

\[
\begin{align*}
\mathbf{c}(n + 1) &= \mathbf{c}(n) + \mu e(n) \sum_q \hat{s}_q \mathbf{z}(n - q)
\end{align*}
\]

\[ e(n) =? \]
Core Eqs. | Dynamics of Error Signal

- \( e(n) =? \)

\[
\begin{align*}
e(n) &= d(n) - \sum_{q} s_q y(n - q) \\
y(n) &= w(n)^T x(n) \\
e(n) &= d(n) - \sum_{q} s_q w(n - q)^T x(n - q) \\
z(n) &= F^T x(n) \\
c(n) &= F^T \{ w(n) - w_{opt} \} \\
e(n) &= e_{opt}(n) - \sum_{q} s_q c(n - q)^T z(n - q)
\end{align*}
\]

\[ e_{opt}(n) = e(n) \bigg|_{w = w_{opt}} \]
Core Eqs. | Dynamics of Error Signal Power

- $J(n) =$?

$$
\begin{align*}
J(n) &= E\{e(n)^2\} \\
e(n) &= e_{opt}(n) - \sum_{q} s_q c(n - q)^T z(n - q)
\end{align*}
$$

& using Independence Assumptions

$$
J(n) = J_{opt} + \sum_{q} s_q^2 E\{c(n - q)^T \Lambda c(n - q)\}
$$

where $J_{opt}(n) = E\{e_{opt}(n)^2\}$. For a stationary noise $J_{opt}(n)$ is time-independent.

$J(n)$ is called the Mean-Square-Error (MSE) function.

$J_{ex}(n)$ is called the excess-MSE function.
Core Eqs. | Towards Standard Models

\[ \mathbf{c}(n + 1) = \mathbf{c}(n) + \mu e(n) \sum_{q} \hat{s}_q \mathbf{z}(n - q) \]

\[ e(n) = e_{opt}(n) - \sum_{q} s_q \mathbf{c}(n - q)^T \mathbf{z}(n - q) \]

\[ J(n) = E\{e(n)^2\} = J_{opt} + J_{ex}(n) \]

\[ J_{ex}(n) = \sum_{q} s_q^2 E\{\mathbf{c}(n - q)^T \mathbf{A} \mathbf{c}(n - q)\} \]

A standard Char. Eq. for EMSE function in z-domain ??

Non-linear model (original Eq. set)
Previous Models | Derivation – By Elliott, Bjarnason and ...

Simplification

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta J_{ex}(n - D) + \alpha J_{opt} \]

\[ \alpha = \mu^2 \sigma_x^4 L \]

\[ \beta = \mu \sigma_x^2 \{2 - \mu \sigma_x^2 (L + 2D)\} \]

Autoregressive Model
Previous Models | Stability Analysis

\[ J_{ex}(n) = \sum_q s_q^2 E\{c(n - q)^T \Lambda c(n - q)\} \]

\[ \Rightarrow J_{ex}(n) = E\{c(n - D)^T \Lambda c(n - D)\} \]

\( J_{ex}(n) \) is a Lyapunov function of system variables. Lyapunov stability requires \( \Delta J_{ex}(n) \) to be negative.

\[ \Delta J_{ex}(n) < 0 \]

\[ \Rightarrow J_{ex}(n + 1) - J_{ex}(n) < 0 \]

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta J_{ex}(n - D) + \alpha J_{opt} \]

\[ \Rightarrow -\beta J_{ex}(n - D) + \alpha J_{opt} < 0 \]

\[ \Rightarrow \beta > 0 \]

\[ \beta = \mu \sigma_x^2 \{2 - \mu \sigma_x^2 (L + 2D)\} \]

\[ \Rightarrow 0 < \mu < \frac{2}{\sigma_x^2 (L + 2D)} \]

\[ \text{Eliott Stability Condition} \]
Previous Models | SS Analysis

When \( n \to \infty \):

\[
J_{\text{ex}}(n + 1) = J_{\text{ex}}(n) = J_{\text{ex}}(n - D) = J_{\text{ex}}(\infty)
\]

\[
J_{\text{ex}}(n + 1) = J_{\text{ex}}(n) - \beta J_{\text{ex}}(n - D) + \alpha J_{\text{opt}}
\]

\[\Rightarrow J_{\text{ex}}(\infty) = J_{\text{ex}}(\infty) - \beta J_{\text{ex}}(\infty) + \alpha J_{\text{opt}}\]

\[\Rightarrow J_{\text{ex}}(\infty) = \frac{\alpha}{\beta} J_{\text{opt}}\]

\[\Rightarrow J_{\text{ex}}(\infty) = \frac{\mu \sigma_x^2 L}{2 - \mu \sigma_x^2 (L + 2D)} J_{\text{opt}}\]

\[
J(n) = J_{\text{opt}} + J_{\text{ex}}(n)
\]

\[\Rightarrow J(\infty) = J_{\text{ex}}(\infty) + J_{\text{opt}}\]

\[\Rightarrow J(\infty) = \frac{\alpha}{\beta} J_{\text{opt}} + J_{\text{opt}}\]

\[M = \frac{J(\infty) - J_{\text{opt}}}{J_{\text{opt}}}
\]

\[\Rightarrow M = \frac{\alpha}{\beta}\]

\[M = \frac{\mu \sigma_x^2 L}{2 - \mu \sigma_x^2 (L + 2D)}
\]

Derived by Bjarnason
Previous Models | CR Analysis

In Lyapunov stability theory, the difference of the Lyapunov function is proportional to the convergence speed.

\[ CR \propto \Delta J_{ex}(n) \]

\[ \Rightarrow CR \propto J_{ex}(n + 1) - J_{ex}(n) \]

Here, we define the convergence rate measure \( \omega \) as:

\[ \omega = \beta \]

\[ \Rightarrow \omega \approx 0 \]

\[ \Rightarrow CR \propto -\beta J_{ex}(n - D) + \alpha J_{opt} \]

Proposed measure

In transient conditions (where the convergence speed is determined), \( J_{ex} \) is greater than \( J_{opt} \):

\[ \Rightarrow CR \propto -\beta J_{ex}(n - D) \]

\[ \Rightarrow \omega = \mu \sigma_{x}^{2}\{2 - \mu \sigma_{x}^{2}(L + 2D)\} \]
ANC Modeling & Analysis

Previous Models | Derivation & Analysis Summary

Propagation: Pure Delay
Noise: Broadband White
Secondary Path Estimate: Perfect
Independence Assumptions

Simplification

\[
J_{ex}(n + 1) = J_{ex}(n) - \beta J_{ex}(n - D) + \alpha J_{opt}
\]

\[
\alpha = \mu^2 \sigma_x^4 L
\]

\[
\beta = \mu \sigma_x^2 \{2 - \mu \sigma_x^2(L + 2D)\}
\]

\[
0 < \mu < \frac{2}{\sigma_x^2(L + 2D)}
\]

\[
M = \frac{\mu \sigma_x^2 L}{2 - \mu \sigma_x^2(L + 2D)}
\]

\[
\omega = \mu \sigma_x^2 \{2 - \mu \sigma_x^2(L + 2D)\}
\]

Simplification

NOT Realistic

Stability Condition

Steady-State Performance

Convergence Rate
ANC Modeling & Analysis

Model 1 | Derivation

- Generalization
- Simplification


\[ J_{ex}(n+1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n-q) + \alpha J_{opt} \]
\[ \alpha = \mu^2 \|s\|^4 \sigma_x^4 L \]
\[ \beta = \mu \sigma_x^2 \{2 - \mu \|s\|^2 \sigma_x^2 (L + 2D_{eq})\} \]

M1 Auto-regressive Model

\[ D_{eq} = \frac{s^T \Psi s}{s^T s} \]
\[ \Psi = diag(0,1,2, \ldots, Q-1) \]

Results have been published in Elsevier Journal of Signal Processing (2010) – [Link]
Model 1 | Derivation | Introducing $D_{eq}$

$$D_{eq} = \frac{s^T \Psi s}{s^T s} = \frac{0 \times s_0^2 + 1 \times s_1^2 + \cdots + q \times s_q^2 + \cdots + (Q - 1) \times s_{Q-1}^2}{s_0^2 + s_1^2 + \cdots + s_q^2 + \cdots + s_{Q-1}^2}$$

For a pure delay secondary path:
$s_0 = s_1 = \cdots = 0$ and $s_D = 1$

In this case, $D_{eq} = D \in \mathbb{N}$

$D_{eq}$ is a physical variable (time-delay in the secondary path).

For an arbitrary secondary path:
$s_q, s_1, \ldots, s_q, \ldots, s_{Q-1}$

In this case, $D_{eq} \in \mathbb{R}$

For example:

$$D_{eq} = \frac{0 \times 0^0 + 1 \times 0.7^2 + 2 \times 0.5^2 + 3 \times 0.1^2}{0^2 + 0.7^2 + 0.5^2 + 0.1^2} = 1.36$$
Model 1 vs Model 0

**M0**

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta J_{ex}(n - D) + \alpha J_{opt} \]

\[ \alpha = \mu^2 \sigma_x^4 L \]

\[ \beta = \mu \sigma_x^2 \{2 - \mu \sigma_x^2 (L + 2D)\} \]

**M1**

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]

\[ \alpha = \mu^2 ||s||^4 \sigma_x^4 L \]

\[ \beta = \mu \sigma_x^2 \{2 - \mu ||s||^2 \sigma_x^2 (L + 2D_{eq})\} \]

Propagation: General (FIR)
Model 1 | Stability Analysis

\[ J_{ex}(n) = \sum_q s_q^2 E \{ c(n - q)^T \Lambda c(n - q) \} \]

\( J_{ex}(n) \) is a Lyapunov function of system variables. Lyapunov stability requires \( \Delta J_{ex}(n) \) to be negative.

\[ \Delta J_{ex}(n) < 0 \]
\[ \Rightarrow J_{ex}(n + 1) - J_{ex}(n) < 0 \]
\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]

\( \Rightarrow -\beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} < 0 \)
\( \Rightarrow \beta > 0 \)

\[ \beta = \mu \sigma_x^2 \{ 2 - \mu \|s\|^2 \sigma_x^2 (L + 2D_{eq}) \} \]
\( \Rightarrow 0 < \mu < \frac{2}{\|s\|^2 \sigma_x^2 (L + 2D_{eq})} \)
Model 1 | SS Analysis

When \( n \to \infty \):

\[ J_{ex}(n + 1) = J_{ex}(n) = J_{ex}(n - q) = J_{ex}(\infty) \]

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]

\[ \Rightarrow J_{ex}(\infty) = J_{ex}(\infty) - \beta \sum s_q^2 J_{ex}(\infty) + \alpha J_{opt} \]

\[ \Rightarrow J_{ex}(\infty) = \frac{\alpha}{|s|^2 \beta} J_{opt} \]

\[ J(n) = J_{opt} + J_{ex}(n) \]

\[ \Rightarrow J(\infty) = J_{ex}(\infty) + J_{opt} \]

\[ \Rightarrow J(\infty) = \frac{\alpha}{|s|^2 \beta} J_{opt} + J_{opt} \]

\[ M = \frac{J(\infty) - J_{opt}}{J_{opt}} \]

\[ \Rightarrow M = \frac{\alpha}{|s|^2 \beta} \]

\[ \Rightarrow M = \frac{\mu |s|^2 \sigma_x^2 L}{2 - \mu |s|^2 \sigma_x^2 (L + 2D_{eq})} \]
Model 1 | CR Analysis

In Lyapunov stability theory, the difference of the Lyapunov function is proportional to the convergence speed.

\[ CR \propto \Delta J_{ex}(n) \]

\[ \Rightarrow CR \propto J_{ex}(n+1) - J_{ex}(n) \]

\[ J_{ex}(n+1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n-q) + \alpha J_{opt} \]

\[ \Rightarrow J_{ex}(n+1) - J_{ex}(n) = -\beta \sum s_q^2 J_{ex}(n-q) + \alpha J_{opt} \]

\[ \Rightarrow CR \propto -\beta \sum s_q^2 J_{ex}(n-q) + \alpha J_{opt} \]

In transient conditions (where the convergence speed is determined), \( J_{ex} \) is greater than \( J_{opt} \):

\[ \Rightarrow CR \propto -\beta \sum s_q^2 J_{ex}(n-q) \]

Here, we define the convergence rate measure \( \omega \) as:

\[ \omega = \beta \]

\[ \Rightarrow \omega = \mu \sigma_x^2 \{ 2 - \mu |s| \sigma_x^2 (L + 2D_{eq}) \} \]
Results have been published in Elsevier Journal of Signal Processing (2010) – [Link]
ANC Modeling & Analysis

Model 2 | Derivation

- Generalization
- Simplification
- Propagation: General (FIR)
- Noise: Band-limited White
- Secondary Path Estimate: Perfect
- Independence Assumptions

\[
J_{ex}(n+1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n-q) + \alpha J_{opt}
\]

\[
\alpha = \mu^2 \|s\|^4 \frac{\sigma_x^4}{B} L
\]

\[
\beta = \mu \frac{\sigma_x^2}{B} \left\{ 2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq}) \right\}
\]

Autoregressive Model

\( B \): normalized noise bandwidth

Results have been published in Elsevier Journal of Signal Processing (2011) – Link
ANC Modeling & Analysis

Model 1 vs Model 2

M1

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]
\[ \alpha = \mu^2 \|s\|^4 \sigma_x^4 L \]
\[ \beta = \mu \sigma_x^2 \{2 - \mu \|s\|^2 \sigma_x^2 (L + 2D_{eq})\} \]

Propagation: General (FIR)

M2

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]
\[ \alpha = \mu^2 \|s\|^4 \frac{\sigma_x^4}{B} L \]
\[ \beta = \mu \frac{\sigma_x^2}{B} \{2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2}{B}D_{eq})\} \]

Noise: Band-limited White

Propagation: General (FIR)
Model 2 | Stability Analysis

Model 2 and Model 1 have the same structure with different scalar coefficients. The stability condition of Model 1 is derived from $\beta > 0$; therefore, The stability condition of Model 2 can be also derived from this inequality:

\[
\beta > 0
\]

\[
\beta = \mu \frac{\sigma_x^2}{B} \left\{ 2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq}) \right\}
\]

\[
\Rightarrow 0 < \mu < \frac{2}{\|s\|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq})}
\]
Model 2 | Stability Analysis | Special Case 2

When the acoustic noise has only one frequency component, the normalized bandwidth can be considered as $\frac{1}{L}$. In this case, the stability condition becomes:

$$0 < \mu < \frac{2}{\|s\|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq})} \Rightarrow 0 < \mu < \frac{2}{\|s\|^2 \sigma_x^2 L(1 + 2D_{eq})}$$

That is identical with the one in ANC literature.
When the acoustic noise is ideally broadband, the normalized bandwidth can be considered as 1. In this case, the stability condition becomes:

\[ 0 < \mu < \frac{2}{\|s\|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq})} \Rightarrow 0 < \mu < \frac{2}{\|s\|^2 \sigma_x^2 (L + 2D_{eq})} \]

That is identical with the one in ANC literature.
Model 2 | SS and CR Analysis

Based on the same logic, the misadjustment level can be formulated.

\[
M = \frac{\alpha}{|s|^2 \beta}
\]

\[
\Rightarrow M = \frac{\mu |s|^2 \sigma_x^2 L}{2 - \mu |s|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq})}
\]

Also, the convergence rate measure is formulated as:

\[
\omega = \beta
\]

\[
\Rightarrow \omega = \mu \frac{\sigma_x^2}{B} \left\{ 2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq}) \right\}
\]
Model 2 | Derivation & Analysis Summary

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]
\[ \alpha = \mu^2 \| s \|^4 \frac{\sigma_x^4}{B} L \]
\[ \beta = \mu \frac{\sigma_x^2}{B} \left\{ 2 - \mu \| s \|^2 \sigma_x^2 (L + 2 \frac{2}{B} D_{eq}) \right\} \]

\[ 0 < \mu < \frac{2}{\| s \|^2 \sigma_x^2 (L + 2 \frac{2}{B} D_{eq})} \]
\[ M = \frac{\mu \| s \|^2 \sigma_x^2 L}{2 - \mu \| s \|^2 \sigma_x^2 (L + 2 \frac{2}{B} D_{eq})} \]
\[ \omega = \mu \frac{\sigma_x^2}{B} \left\{ 2 - \mu \| s \|^2 \sigma_x^2 (L + 2 \frac{2}{B} D_{eq}) \right\} \]

Stability Condition
Steady-State Performance
Convergence Rate

Results have been published in Elsevier Journal of Signal Processing (2011) – Link
Model 3 | Derivation

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum_{q} s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]

\[ \alpha = \rho_1 \mu^2 \|s\|^4 \frac{\sigma_x^4}{B} L \]

\[ \beta = \rho_1 \mu \frac{\sigma_x^2}{B} \left\{ 2 \rho_2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2 \rho_2 \rho_3}{B} D_{eq}) \right\} \]

\[ \rho_1, \rho_2 \text{ and } \rho_3 \text{ are perfectness ratios.} \]

\[ \rho_1 = \frac{\|\hat{s}\|^2}{\|s\|^2} \quad \rho_2 = \frac{s^T \hat{s}}{\|\hat{s}\|^2} \quad \rho_3 = \frac{s^T \Psi \hat{s}}{s^T \Psi s} \]

\[ \Psi = \text{diag}(0,1,2,\ldots,Q-1) \]

For a perfect estimate model (\( \hat{s} = s \)):

\[ \rho_1 = \rho_2 = \rho_3 = 1 \]
Model 2 vs Model 3

**M2**

\[
J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt}
\]

\[
\alpha = \mu^2 \|s\|^4 \frac{\sigma_x^4}{B} L
\]

\[
\beta = \mu \frac{\sigma_x^2}{B} \left\{ 2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2}{B} D_{eq}) \right\}
\]

- Noise: Band-limited White
- Propagation: General (FIR)

**M3**

\[
J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt}
\]

\[
\alpha = \rho_1 \mu^2 \|s\|^4 \frac{\sigma_x^4}{B} L
\]

\[
\beta = \rho_1 \mu \frac{\sigma_x^2}{B} \left\{ 2 \rho_2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2 \rho_2 \rho_3}{B} D_{eq}) \right\}
\]

- Secondary Path Estimate: Arbitrary
- Noise: Band-limited White
- Propagation: General (FIR)
Model 3 | Stability Analysis

Model 3 and Model 2 (& 1) have the same structure with different scalar coefficients. The stability condition of Model 2 is derived from $\beta > 0$; therefore, The stability condition of Model 3 can be also derived from this inequality:

$$\beta > 0$$

$$\beta = \rho_1 \mu \frac{\sigma_x^2}{B} \left\{2 \rho_2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2 \rho_2 \rho_3}{B} D_{eq})\right\}$$

$$\Rightarrow 0 < \mu < \frac{2 \rho_2}{\|s\|^2 \sigma_x^2 (L + \frac{2 \rho_2 \rho_3}{B} D_{eq})}$$

For having $\mu > 0$; it is essential to hold $\rho_2 > 0$.

$$\rho_2 = \frac{s^T \hat{s}}{\|\hat{s}\|^2}$$

$$\Rightarrow s^T \hat{s} > 0 \Rightarrow \theta(s, \hat{s}) < 90^\circ$$
Model 3 | SS and CR Analysis

Based on the same logic, the misadjustment level can be formulated.

\[
M = \frac{\alpha}{|s|^2 \beta}
\]

\[\Rightarrow M = \frac{\mu |s|^2 \sigma_x^2 L}{2 \rho_2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2 \rho_2 \rho_3}{B} D_{eq})}\]

Also, the convergence rate measure is formulated as:

\[
\omega = \beta
\]

\[\Rightarrow \omega = \rho_1 \mu \frac{\sigma_x^2}{B} \left\{2 \rho_2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2 \rho_2 \rho_3}{B} D_{eq})\right\}\]
 ANC Modeling & Analysis

Model 3 | Derivation & Analysis Summary

\[ J_{ex}(n + 1) = J_{ex}(n) - \beta \sum s_q^2 J_{ex}(n - q) + \alpha J_{opt} \]

\[ \alpha = \rho_1 \mu^2 \frac{\|s\|^4 \sigma_x^4}{B} \]

\[ \beta = \rho_1 \mu \frac{\sigma_x^2}{B} \left\{ 2\rho_2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2\rho_2 \rho_3}{B} D_{eq}) \right\} \]

0 < \mu < \frac{2\rho_2}{\|s\|^2 \sigma_x^2 (L + \frac{2\rho_2 \rho_3}{B} D_{eq})}

\[ M = \frac{\mu \|s\|^2 \sigma_x^2 L}{2\rho_2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2\rho_2 \rho_3}{B} D_{eq})} \]

\[ \omega = \rho_1 \mu \frac{\sigma_x^2}{B} \left\{ 2 - \mu \|s\|^2 \sigma_x^2 (L + \frac{2\rho_2 \rho_3}{B} D_{eq}) \right\} \]

Stability Condition

Steady-State Performance

Convergence Rate

Results have been published in IEEE (2012) and IET Journals (2013) – [Link (IEEE)] / [Link (IET)]
ANC Modeling & Analysis

Models 1, 2 and 3 | Validation

• Analytical

  \[ \rho_1, \rho_2, \rho_3 = 1 \quad B = 1 \quad \text{Pure-delay} \]

• Computer Simulation
  - Reported in our publications as well as in my PhD thesis.

• Practical Experiments
  - Reported in our publications as well as in my PhD thesis.
ANC Modelling & Analysis

• ANC Modelling & Analysis ✓
• ANC Adaptation Process Dynamic Control

Questions?
Part II – Recent Advances

ANC
Dynamic Control
Content

- Motivation
- Dynamics of ANC Adaptation Process
- Root Locus Analysis
- FwFxLMS Algorithm
Motivation
Improving robustness of ANC systems

Traditional ANC algorithms suffer low robustness in real-life applications.

ANC Dynamic Control

System robustness

Uncertainty level in environmental conditions

High
Moderate
Low

1
2
3

158
ANC Modeling & Analysis

Dynamics of ANC Adaptation Process

\[ x(n) \] statistics \rightarrow ANCs Models [Type 1] \rightarrow \[ e(n) \] statistics

- Step size and secondary path model

Suitable for understanding ANC behaviour in \textit{Physical Domain (Acoustic)}

\[ x(n) \] statistics \rightarrow ANCs Models [Type 2] \rightarrow \[ W(n) \] statistics

- Step size and secondary path model

Suitable for understanding ANC behaviour in \textit{Control Domain (Electronics)}
Dynamics of ANC Adaptation Process

\[
\begin{align*}
\mathbf{c}(n+1) &= \mathbf{c}(n) + \mu e(n) \sum_{\hat{q}} \hat{s}_{\hat{q}} \mathbf{z}(n - \hat{q}) \\
e(n) &= e_{\text{opt}}(n) - \sum_{q} s_q \mathbf{c}(n - q)^T \mathbf{z}(n - q)
\end{align*}
\]

\[
\Rightarrow \mathbf{c}(n+1) = \mathbf{c}(n) + \mu \left( e_{\text{opt}}(n) - \sum_{q} s_q \mathbf{c}(n - q)^T \mathbf{z}(n - q) \right) \sum_{\hat{q}} \hat{s}_{\hat{q}} \mathbf{z}(n - \hat{q})
\]

\[
\Rightarrow \mathbf{c}(n+1) = \mathbf{c}(n) + \mu \sum_{\hat{q}} \hat{s}_{\hat{q}} \mathbf{z}(n - \hat{q}) e_{\text{opt}}(n) - \mu \sum_{q, \hat{q}} s_{\hat{q}} \hat{s}_{p} \mathbf{z}(n - \hat{q}) \mathbf{z}(n - q)^T \mathbf{c}(n - q)
\]
Dynamics of ANC Adaptation Process

\[ \Rightarrow c(n + 1) = c(n) + \mu \sum_{\hat{q}} \hat{s}_{\hat{q}} z(n - \hat{q}) e_{opt}(n) - \mu \sum_{q, \hat{q}} s_{\hat{q}} \hat{s}_p z(n - \hat{q}) z(n - q)^T c(n - q) \]

\[ \Rightarrow \bar{c}(n + 1) = \bar{c}(n) + \mu \sum_{\hat{q}} \hat{s}_{\hat{q}} E\{ z(n - \hat{q}) e_{opt}(n) \} - \mu \sum_{q, \hat{q}} s_{\hat{q}} \hat{s}_p E \{ z(n - \hat{q}) z(n - q)^T c(n - q) \} \]

2\text{nd IA} \quad 2^{nd} \text{IA} \quad 0

3\text{rd IA} \quad 3^{rd} \text{IA} \quad \mu \sum_{q, \hat{q}} s_{\hat{q}} \hat{s}_p E \{ z(n - \hat{q}) z(n - q)^T \} \bar{c} (n - q)

1\text{st IA} \quad 1^{st} \text{IA} \quad \mu \sum_{q} s_{q} \hat{s}_q \Lambda \bar{c} (n - q)

\Rightarrow \bar{c}(n + 1) = \bar{c}(n) - \mu \sum_{q} s_{q} \hat{s}_q \Lambda \bar{c} (n - q) \Rightarrow \bar{c}(n + 1) - \bar{c}(n) + \mu \sum_{q} s_{q} \hat{s}_q \Lambda \bar{c} (n - q) = 0
Dynamics of ANC Adaptation Process

\[
\Rightarrow \bar{c}(n + 1) - \bar{c}(n) + \mu \sum_{q} s_q \hat{s}_q \Lambda \bar{c} (n - q) = 0
\]

For a perfect secondary path estimate
\[ s_q = \hat{s}_q \]

\[
\Rightarrow \bar{c}(n + 1) - \bar{c}(n) + \mu \sum_{q} s_q^2 \Lambda \bar{c} (n - q) = 0
\]

For an ideal white noise
\[ \Lambda = \sigma_x^2 I \]

\[
\Rightarrow \bar{c}(n + 1) - \bar{c}(n) + \mu \sigma_x^2 \sum_{q} s_q^2 \bar{c} (n - q) = 0
\]

z transform

\[
\Rightarrow z \bar{C} - \bar{C} + \mu \sigma_x^2 \sum_{q} s_q^2 z^{n-q} \bar{C} = 0 \quad \Rightarrow \quad z - 1 + \mu \sigma_x^2 \sum_{q} s_q^2 z^{n-q} = 0
\]
Dynamics of ANC Adaptation Process

\[ z - 1 + \mu \sigma_x^2 \sum_q s_q^2 z^{-q} = 0 \]

\[ 1 + \mu \sigma_x^2 \frac{\sum_q s_q^2 z^{-q}}{z - 1} = 0 \]

\[ 1 + \mu \sigma_x^2 \frac{\sum_q s_q^2 z^{Q-1-q}}{z^{Q-1} (z - 1)} = 0 \]

FxLMS Adaptation Process
Characteristic Equation

In standard form:

\[ 1 + \mu \sigma_x^2 H(z) = 0 \]

\[ H(z) = \frac{\sum_q s_q^2 z^{Q-1-q}}{z^{Q-1} (z - 1)} \]

Close loop system characteristic Eq
Open loop transfer function
Root Locus Analysis

• Every linear system with a characteristic equation in the form of $1 + \beta H(z) = 0$ can be analyzed using Root Locus Method.
• For a given open loop transfer function, root locus gives the locations of the closed loop system poles while the scalar coefficient $\beta$ is varies.

\[
1 + \mu \sigma_x^2 H(z) = 0
\]

\[
H(z) = \frac{\sum q s_q^2 z^{Q-1-q}}{z^{Q-1} (z - 1)}
\]
Root Locus Analysis | Open Loop Poles & Zeros

\[ H(z) = \frac{\sum_q s_q 2z^{Q-1-q}}{z^{Q-1}} \times \frac{z^{Q-1}}{(z - 1)} = \frac{s_0 2z^{Q-1} + s_1 2z^{Q-2} + \cdots + s_{Q-1}^2}{z^{Q-1}} \]

\[ s = [s_0, s_1, \ldots, s_{Q-1}] \] is the impulse response vector of the secondary path. The first \( Q_0 \) coefficients may be zero:

\[ s = [s_0, s_1, \ldots, s_{Q_0-1}, s_{Q_0}, \ldots, s_{Q-1}] \]

Therefore, \( H(z) \) can be represented by

\[ H(z) = \frac{s_{Q_0} 2z^{Q-1-Q_0} + \cdots + s_{Q-1}^2}{z^{Q-1}} \times \frac{z^{Q-1}}{(z - 1)} \]

\[ N_{\text{zeros}} = Q - 1 - Q_0 \]

\[ N_{\text{poles}} = Q \]
Root Locus Analysis | End Points

Root Locus Theory

\( N_{zeros} \) branches of the FxLMS root locus end at the zeros of the open loop transfer function and the rest of them

\( (N_{poles} - N_{zeros} \) branches) approach infinity (asymptotes)

FXLMS Adaptation Process Root Locus

\[ N_{poles} = Q \]
\[ N_{zeros} = Q - 1 - Q_o \]

\( Q - 1 - Q_o \) branches end at the finite roots and \( Q_o + 1 \) branches approach infinity (asymptotes)
### Root Locus Theory

The number of branches of a root locus plot is equal to the number of poles of the open loop transfer function, $N_{poles}$.

The branches of the root locus starts from the poles of the open loop transfer function.

### FXLMS Adaptation Process Root Locus

$N_{poles} = Q$; thus, the FxLMS root locus has $Q$ branches:

$$B_1, B_2, ..., B_Q$$

$$H(z) = 0 \Rightarrow z^{Q-1} (z - 1) = 0$$

$$\Rightarrow \begin{cases} z_1 = 1 \\ z_2, ..., z_{Q-1} = 0 \end{cases}$$

$B_1$ starts from $z = 1$ and $B_2, ..., B_Q$ start from $z = 0$. 
Root Locus Analysis | Asymptotes

Root Locus Theory
Asymptotes meet at a certain location on the real-axis, given by

$$x_A = \frac{\sum \text{poles of } H(z) - \sum \text{zeros of } H(z)}{N_{\text{poles}} - N_{\text{zeros}}}$$

They also form the following angles

$$\varphi_k = \frac{(2k + 1)\pi}{N_{\text{poles}} - N_{\text{zeros}}}$$

$$k = 0, 1, ..., N_{\text{poles}} - N_{\text{zeros}} - 1$$

FXLMS Adaptation Process Root Locus

$$\sum \text{poles of } H(z) = 1$$

and

$$\sum \text{zeros of } H(z) = -\frac{s_{Q_o+1}^2}{s_{Q_o}^2}$$

Because zeros of $H(z)$ are roots of

$$s_{Q_o}^2 z^{Q_o-1} + s_{Q_o+1}^2 z^{Q_o-2} + ... + s_{Q-1}^2 = 0$$

Thus,

$$x_A = \frac{1 + \frac{s_{Q_o+1}^2}{s_{Q_o}^2}}{Q_o + 1}$$

$$\varphi_k = \frac{(2k + 1)\pi}{Q_o + 1}, k = 0, 1, ..., Q_o$$
Root Locus Analysis | Example

\[ s = [0, 0, 1, 1] \quad Q = 4 \quad Q_0 = 2 \]

\[ H(z) = \frac{z + 1}{z^3(z - 1)} \]

\[ N_{\text{poles}} = Q = 4 \Rightarrow 4 \text{ branches} \]

\[ N_{\text{zeros}} = Q - 1 - Q_0 = 1 \]

\[ x_A = \frac{1 + 1}{3} = 0.667 \]

\[ \varphi_0 = \frac{\pi}{3}, \varphi_1 = \pi, \varphi_2 = \frac{5\pi}{3} \]

\( \text{x: start points} \quad \text{o: end points} \)
### Root Locus Theory
Any parts of the real axis with an odd number of open-loop poles and zeros to the right of them are part of the root locus.

<table>
<thead>
<tr>
<th>FXLMS Adaptation Process Root Locus</th>
</tr>
</thead>
<tbody>
<tr>
<td>The only interval on the positive real axis, which belongs to the FxLMS root locus, is (0,1).</td>
</tr>
<tr>
<td>The root locus plot has no other real segment on the positive real axis but it may have segments on the negative real axis.</td>
</tr>
</tbody>
</table>
Root Locus Analysis | Breakaway Point

Root Locus Theory
The root locus breakaway points are located at the roots of

\[
\frac{\partial}{\partial z} \left\{ \frac{1}{H(z)} \right\} = 0
\]

FXLMS Adaptation Process Root Locus
The FxLMS root locus has always a breakaway point in the real axis between 0 and 1 at the location of

\[
x_B = \frac{D_{eq}}{D_{eq} + 1}
\]

\[
D_{eq} = \frac{0 \times s_0^2 + 1 \times s_1^2 + \cdots + (Q - 1) \times s_{Q-1}^2}{s_0^2 + s_1^2 + \cdots + s_{Q-1}^2}
\]
Example (continued): $s = [0,0,1,1]$

\[
\frac{\partial}{\partial z} \left\{ \frac{1}{H(z)} \right\} = 0 \Rightarrow x_B = 0.71, -1.38
\]

\[
D_{eq} = \frac{0 \times 0^2 + 1 \times 0^2 + 2 \times 1^2 + 3 \times 1^2}{0^2 + 0^2 + 1^2 + 1^2} = 2.5
\]

\[
x_B = \frac{2.5}{2.5 + 1} = 0.71
\]

When $\beta = 0$ poles of the closed loop system are located at the start points, they move on the branches while $\beta$ increases.

When all the poles are inside the unit circle, the system is stable.
Root Locus Analysis | MATLAB Simulation

\[
H(z) = \frac{s_0^2 z^{Q-1} + s_1^2 z^{Q-2} + \cdots + s_{Q-1}^2}{z^{Q-1} (z - 1)}
\]

>> num=[s_0 \ s_1 \ s_2 \ldots \ s_{Q-1}]^2;
>> den=[1 -1 \text{zeros(1,Q-2)}];
>> k=0:0.001:1;
>> rlocus(num,den,k);
Root Locus Analysis | Dominant Pole

For any secondary path,

- $B_1$ starts by getting away from the critical point but it turns back towards the unit circle after reaching $x_B$.
- Other branches start moving from the origin.
- Therefore, $B_1$ always contains the closest points to the critical points (dominant pole).

Results have been published in Journal of Acoustical Society of America (2011) - [Link](#)
Can we remove $x_B$ from the root locus so that the dominant pole can get closer to the origin (improving stability margin and robustness)?

By locating a zero on behind $x_B$ on the real axis.
Root Locus Analysis | Dominant Pole Localisation

• Example:
FwFxLMS Algorithm

By filtering the weight vector using a recursive filter, a real zero located between 0 and 1 can be created for the FxLMS root locus.

\[
\mathbf{w}(n + 1) = \mathbf{w}_a(n) + \mu e(n) \sum_q \hat{s}_q \mathbf{x}(n - q)
\]

\[
\mathbf{w}_a(n) = (1 - \xi)\mathbf{w}(n) + \xi \mathbf{w}_a(n) - 1
\]

\[
2 - x_B - 2\sqrt{1 - x_B} < \xi < 1
\]

Results have been published in Journal of Acoustical Society of America (2013) - [Link](#)
FwFxLMS Algorithm | Results

FxLMS: $\xi = 0$
FwFxLMS: $\xi = 0.7$
FwFxLMS: $\xi = 0.2$
FwFxLMS: $\xi = 0.6$

FxLMS-based ANC
Bibliography (1/2)


Bibliography (2/2)


Citation

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